國立中正大學九十四學年度學士班二年級轉學生招生考試試題

學系別：數學、地球與環境科學、物理、化學暨生物化學、資訊工程、機械工程、經濟學系　　科目：微積分

Calculus

Part I (20%) - Multiple choice problems

5% each. NO partial credits.

(1) $\int_0^\infty x^n e^{-x}dx = ?$ (n is a positive integer)

(a) $n!$ (b) e^n (c) $(n-1)!$ (d) e^{n+1} (e) diverges

(2) Suppose $f(x)$ is twice-differentiable. Then $\lim_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{2h^2} = ?$

(a) $f'(x)$ (b) $\frac{f'(x)}{2}$ (c) $\frac{f'(x+)+f'(x-)}{2}$ (d) $\frac{f''(x)}{2}$ (e) $f''(x)$

(3) If $f(x,t) = \sin(2x+3t)$ is a solution of the wave equation $\frac{\partial^2 f}{\partial t^2} - c^2\frac{\partial^2 f}{\partial x^2} = 0$, then $c = ?$

(a) $\frac{4}{9}$ (b) $\frac{3}{2}$ (c) $\frac{2}{3}$ (d) 1 (e) $\frac{9}{4}$

(4) Which is not a relative (local) extreme point of the function $f(x,y) = xye^{-(x^2+y^2)/2}$?

(a) $(1,1)$ (b) $(1,-1)$ (c) $(-1,1)$ (d) $(-1,-1)$ (e) $(0,0)$

Part II (50%) - Fill in the blanks

10% each. NO partial credits.

(1) Let $x(t) = \sin^2 t$ and $y(t) = \cos t$. Then, when $t = \frac{1}{4}\pi$, $\frac{d^2y}{dx^2} =$ ______

(2) In the following four series,

(a) $\sum_{k=2}^{\infty} \frac{1}{k\ln k}$ (b) $\sum_{n=2}^{\infty}\left(\frac{1}{\sqrt{n}-1} - \frac{1}{\sqrt{n}+1}\right)$ (c) $\sum_{n=1}^{\infty}\frac{1}{2^{\ln n}}$ (d) $\sum_{k=1}^{\infty}\frac{1+\sin k}{(k+1)^{3/2}}$

______ of them converge(s), i.e, how many of them are convergent ?

(3) The minimum of $f(x,y,z) = 4x-2y+3z$ subject to the constraint $2x^2+y^2-3z=0$ is ______.

(4) If the vector function $\frac{x}{y^2}\mathbf{i} - \left(\frac{x^2}{y^3} + y^2\right)\mathbf{j}$ is the gradient $\nabla f(x, y)$ of the function $f(x, y)$, then $f(x, y) =$ ______.

(Answer "None" if no such f exists.)

(5) Let Ω the triangle formed by the y-axis, $2y = x$, $y = 1$.

Then $\iint\limits_{\Omega} e^{-y^2/2} dxdy =$ ______

PART III (30%) - COMPUTATIONAL PROBLEMS

Show all your work. NO CREDITS if only present answers.

(1) Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$ be a smooth curve defined by twice-differentiable functions $x = f(t)$ and $y = g(t)$.

(a) (8%) Show that the curvature κ of the curve is given by the formula

$$\kappa = \frac{|f'g'' - g'f''|}{[(f')^2 + (g')^2]^{3/2}}.$$

(b) (7%) Let $\mathbf{r}$ be the ellipse $\mathbf{r}(t) = a\cos t\mathbf{i} + b\sin t\mathbf{j}$ with $a > b > 0$. Show that $\mathbf{r}$ has its largest curvature on its major axis and its smallest curvature on its minor axis.

(2) Find the Jordan curve C that maximizes the line integral (8%) and the corresponding maximum (7%).

$$\oint_C (y^3 + e^{-x^2})dx + (e^{-y^2} - x^3 + 6xy)dy.$$