1. (25%)Please provide the general solutions of the following differential equations:

   (a) (8%) $y''-8y'+16y=0$

   (b) (8%) $y''-y'-12y=0$

   (c) (9%) $y''-6y'+10y=0$

2. (15%)

   (a) Fitting a straight line to a set of points: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ can be done by the linear least-squares regression. Let $y = a_0 + a_1 x$ be the fitted straight line. (5%)

   Derive the solution of a_0 and a_1.

   (b) Show that the problem of (a) can also be solved by the general least-squares solution (5%)

$$\vec{a} = (A^T A)^{-1} A^T \vec{y}$$

   Where $\vec{a} = [a_0, a_1]^T$ *and* $\vec{y} = [y_1, y_2, \ldots, y_n]^T$.
   What is the content of $[A^T A]$?

   (c) If $y = a_0 + a_1 x + a_2 x^2$, what is the content of A? (5%)

3. (10%) $A\vec{x} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 5 & 1 \\ -1 & 3 & a \end{bmatrix} \vec{x} = \begin{bmatrix} 3 \\ 5 \\ b \end{bmatrix}$

   (a) If $\vec{x}$ has more than 1 solution, find a and b. (5%)

   (b) Find $\overrightarrow{x_h}$ and $\overrightarrow{x_p}$ such that $\overrightarrow{x_g} = \overrightarrow{x_h} + \overrightarrow{x_p}$ *and* $A\overrightarrow{x_g} = \begin{bmatrix} 3 \\ 5 \\ b \end{bmatrix}$, where $\overrightarrow{x_g}$, $\overrightarrow{x_h}$, $\overrightarrow{x_p}$, are general solution, homogeneous solution, and particular solution respectively. (5%)

4. (10%) Find the Laplace transform F(s) of the function f(t)

   (a) (5%) $f(t)=e^{-2t}$

   (b) (5%) $f(t)=e^{-t}\cos(10t)$

5. (15%) Given the following differential equation

$$y'' - 2y' - 3y = f(t), y(0) = 1, y'(0) = 0$$

   Please solve the problems using the Laplace transform approach

   (a) (5%) please solve y(t) with the above initial conditions and the input f(t)=0

   (b) (5%) please solve y(t) with the above initial conditions with the input f(t) given as

$$f(t) = 0, t < 0$$
$$f(t) = 3, t \geq 0$$

   (c) (5%) Please solve y(t) with the above initial conditions with input f(t) given as

$$f(t) = 0, t < 4$$
$$f(t) = 3, t \geq 4$$

6. (25%) Free vibrations of a string installed on a guitar are described by the wave equation as shown below:

$$T\frac{\partial^2 w(x,t)}{\partial x^2} = \rho_o A\frac{\partial^2 w(x,t)}{\partial t^2}$$

   where T is the string tension, ρ_o is the string density and A is the string cross sectional area. If this string has a length L and fixed at both ends, i.e. $w(x{=}0, t){=}0$ and $w(x{=}L, t){=}0$. Please determine

   (a) (8%) the eigenvalue,

   (b) (7%) the eigenfunction and

   (c) (10%) the string displacement $w(x, t)$ if the initial conditions are given by

$$w(x,0) = \sin\frac{\pi x}{L} \quad \text{and} \quad \frac{\partial w}{\partial t}(x, t=0) = 0$$

Hint: $\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \begin{cases} L/2 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$