1. (10%) A mass-spring-damper system $M\ddot{y}(t)+C\dot{y}(t)+Ky(t)=r(t)$ is with a unit mass and unknown values of spring constant K and damping coefficient C. A unit impulse function $r(t)=\delta(t)$ generates an output response as $y(t)=e^{-t}-e^{-2t}$. Now if we are given another input function $r(t)=\sin t$, please find the corresponding output response.

2. (5%) Find the inverse Laplace transform of $\dfrac{4}{s^2+4s+20}$

3. (10%) Solve the following system of differential equations:

$$2\frac{dx}{dt}+\frac{dy}{dt}-y=t$$
$$\frac{dx}{dt}+\frac{dy}{dt}=t^2$$

with the initial condition of $x(0)=1$ and $y(0)=0$

4. (15%) For a function $\varphi = x - 2xy + yz^2$, determine:

(4a) (3%) The gradient of function φ at point P (3, 1, -2).

(4b) (2%) What is the physical (or geometrical) meaning of the gradient obtained in (4a)?

(4c) (3%) Find the curl of the gradient at point P obtained in (4a).

(4d) (5%) Find the directional derivative of the function φ at point P in the direction of vector $\mathbf{v} = 7\,\mathbf{i} + 6\,\mathbf{j} + 6\,\mathbf{k}$.

(4e) (2%) What is the physical meaning of the directional derivative obtained in (4d)?

5. (10%) Consider a matrix $A=\begin{pmatrix} 15 & 6 & -12 \\ 4 & 10 & -2 \\ -4 & 8 & -7 \end{pmatrix}$.

(5a) (2%) Find the rank of the matrix A.

(5b) (5%) Determine the characteristic values and the characteristic vectors of the matrix A.

(5c) (3%) Diagonalize the matrix A.

6. (25%) Considering an one-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

with boundary conditions:

$$u(0,t) = 0, \quad u(L,t) = 0 \quad \text{for all t} \tag{2}$$

and initial conditions:

$$u(x,0) = f(x), \ \frac{\partial u(x,0)}{\partial t} = g(x) \tag{3}$$

(6a) (8%) Using method of separating variables $[u(x,t) = F(x)G(t)]$, please show that Eq. (1) can yield the following two ordinary differential equations.

$$\frac{d^2 F}{dx^2} - kF = 0$$

$$\frac{d^2 G}{dt^2} - c^2 kG = 0 \tag{4}$$

where k is a constant.

(6b) (12%) Applying the boundary and initial conditions [i.e. Eqs. (2) and (3)], please show that the solution of the one-dimensional wave equation is

$$u(x, t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + B_n^* \sin \lambda_n t) \sin \frac{n\pi}{L} x \tag{5}$$

where

$$\lambda_n = \frac{cn\pi}{L}, \qquad B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx, \qquad B_n^* = \frac{2}{cn\pi}\int_0^L g(x) \sin\frac{n\pi x}{L} dx$$

(6c) (5%) Please find the deflection $u(x,t)$ of a vibrating string ($L = \pi,\ c^2 = 1$) corresponding to zero velocity and initial deflection: $f(x) = 5\sin 3x$.

7. (15%) Consider the following periodic function f(x) with a period of 2π:

$$f(x)=\begin{cases}1 & \text{if } -\pi/2<x<\pi/2 \\ 0 & \text{if } \pi/2<x<3\pi/2\end{cases}$$

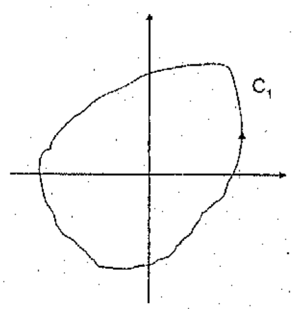
7a. (10%) Find its Fourier series.

7b. (5%) By calculating $\int_{-\pi}^{\pi}|f(x)|^2 dx$, show that

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}$$

8. (5%) Show that the complex function $f(z)=z^*$ is not analytic where z^* denotes the complex conjugate of z.

9. (5%) Let C_1 be an arbitrary closed path containing the origin of the complex plane as shown in the figure below.



Knowing that $\int_{C_0}\frac{1}{z}dz=2\pi i$ over the unit circle C_0 in the counterclockwise sense, show that $\int_{C_1}\frac{1}{z}dz=2\pi i$ over C_1 by using the Cauchy's integral theorem.